

Variance-based control of regime shifts: bistability and oscillations

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Abstract

A variety of real world and experimental systems can display a drastic regime shift, as the evolution in one of its parameters crosses a threshold value. Assimilation of such a transition with a bifurcation has allowed to identify so called “early warning signals”, at the level of the time series generated by the system under scope. The literature in early warning detection methods is currently expanding and their potential for practical applicability is being discussed in different contexts. In this work, we elaborate on the use of the variance of a system variable, which constitutes the simplest early warning indicator, to gain control on the long-term dynamics of the system, while extending an exploitation phase. In particular, we address the cases of the cusp and Hopf normal forms, as prototypical examples of bistability and oscillations. Our results provide insights on the interplay between the time-scale for the system observation, the degree of sensitivity of the control feedback and the intensity of the random perturbations, in shaping the long-term control efficiency.

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I. INTRODUCTION

Along the last few decades an increasing number of scholars has been contributing to develop a coherent and further integral body of knowledge on complex real world systems. Cross-fertilization of concepts and tools stemming from dynamical systems, statistical physics and information and computation theories has allowed to identify and analyse systems whose observable behaviors result from similar underlying mechanisms [1, 2]. As a result of these attempts, a key finding has been realization of the ubiquity of nonlinear feedbacks in the dynamics of different classes of systems.

In particular, the occurrence of drastic, qualitative shifts of the functioning regime of systems—as resulting from strong positive and negative feedbacks—has attracted interest in diverse research fields, as diverse as microscopic physical systems at equilibrium [3] and their macroscopic non-equilibrium counterpart [4, 5], ecology [6, 7], socio-physics [8] and Earth system science [7, 9].

Insights into the phenomenology, related to regime shifts and the development of new methods for their early detection, have relied on minimal models. A frequent idea in this approach is to cast the description of the system evolution as a stochastic differential equation [10], of the form:

$$d\mathbf{x} = \mathbf{F}(\mathbf{x}, \lambda)dt + d\mathbf{W}_x \quad (1)$$

The key variables of the system are represented as a vector $\mathbf{x} = (x_1, x_2, \dots, x_N)$ and their dynamic interrelationships are considered as deterministic laws, described as a vector field \mathbf{F} . Vector \mathbf{W}_x aims at introducing the effect of incessant perturbations on the \mathbf{x} components, as typically resulting from endogenous meso-scale dynamic complexity and from an infinite number of uncorrelated environmental factors. Consequently, \mathbf{W}_x is assimilated to a Gaussian Wiener process [10] with standard deviation ω_x . The parameter λ controls the qualitative behavior of solutions to the deterministic part of system (1) around a stationary state \mathbf{x}_s . In this framework, a regime shift is represented as a local bifurcation occurring in the deterministic part of the system at a critical point $(\mathbf{x}_s, \lambda_c)$, i.e., that the Jacobian matrix of \mathbf{F} , evaluated at \mathbf{x}_c , possesses one eigenvalue whose real part tends to zero as λ approaches the value λ_c [11]. Thereby, the response of the system to random fluctuations can be studied along the evolution towards the bifurcation point.

Given the relevance of applications and implications of regime shifts in real world systems, it is of increasing interest to construct early warning indicators (**EWI**) [12, 13] – with focus on identifying statistical signatures of the loss of stability in the time series generated by the evolution of the system towards criticality. Conjointly, a relevant phenomenon is the slowing down of the

fast stabilizing dynamic modes –often referred as critical slowing down. This results from the real part of the leading eigenvalue becoming zero as the system closely approaches the bifurcation point [12].

As illustrated in a pioneering work by S. Carpenter [14] on **EWI**-based monitoring of shallow lake ecosystems, the onset of the critical slowing down can lead to a drastic increase in the variance of a variable of the system. More recently, the skewness [15], kurtosis [16] and the autocorrelation function [17] of a system variable have been shown to capture relevant information on the increased loss of stability close to the bifurcation point.

The literature in early warning detection is rapidly expanding and their potential applications within a broad range of real world problems is being discussed [12, 13]. Within this contextual frame, it becomes natural to inquire about the possible long-term complex behavior that would arise if **EWI**s are repeatedly used to control systems potentially undergoing regime shifts. As far as we know, this question remains open, in the light of earlier and further recent advanced methods in the construction of **EWI**'s [18].

In order to tackle this question, we propose to analyze the dynamics of a system represented by (1) subject to a control process. The basic idea is that such a steering control should: 1) re-establish the stability of the system before it crosses a critical point, or 2) re-establish a lost initial system regime once a critical point has been crossed, to then let the system evolve again towards criticality.

Let us explore this idea by considering a control system which reacts upon the behavior of an **EWI**, denoted by I_τ , whose construction is based on information about the past states of the system within a sliding observation time window of size τ . In other words, we are interested in analyzing the coupled dynamics of (1) with an evolution equation for λ , of the form

$$d\lambda = \Phi(I_\tau, \alpha)dt + dW_\lambda \quad (2)$$

Here we require the control functional Φ to operate a change of sign, at the level of $d\lambda$, once I_τ crosses a reference value V_R . In order to allow for differences in the degree of sensitivity of the control system, let us consider the control response to changes in the ratio I_τ/V_R as modulated by a parameter α . Moreover, let us assume that the control response may exhibit random variability as a result of the influence of many uncorrelated factors. Thus, we denote by W_λ a Gaussian Wiener process with standard deviation ω_λ .

In the domain of application of **EWI**s, relevant cases are those where the control variable λ is driven by human interventions. In many of these instances, the evolution of the system towards

criticality is the result of an “exploitation” process, that is sought to be intensified and maintained, while avoiding a system regime shift. Consider for instance catching rates in sustainable fisheries [19], agriculture related leaching rates of phosphorus into a shallow lake, whose oligotrophic regime is to be preserved [20], or the increase in the autocatalytic production of chemical species in a well-stirred reactor [21], where a low entropy production regime may be desirable. Consequently, we characterize the dynamics of system (1)-(2) in terms of two phases: *exploitation* if $\Phi > 0$ and *recovery* if $\Phi < 0$.

Central questions for us are:

- i) The role of the time-scale τ , considered for the construction of the **EWI**s, in shaping the emergence of control properties in a system (1)-(2).
- ii) The long-term complex dynamics that can result from the temporal non-locality introduced by the **EWI** into the dynamics of the coupled system (1)-(2). In other words, in the long-term, the control exerts changes on the system on the basis of the past dynamics of the system and its control as a whole. This opens the possibility to observe different complex behaviors, as the effective dimension of the system-control is augmented by its dependency on the past.
- iii) The efficiency of the control process in terms of: its capacity to increase and maintain exploitation in a selected regime as a function of the observation window τ , of the control response sensitivity α and of the variance of stochastic perturbations ω_x and ω_λ .

In Sec.II we address these questions, by exploring numerically the performance of control on minimal dynamical systems of generic character, that reproduce the prototypical phenomena of bistability and oscillations. Each of these controlled systems is first studied in absence of noise, in order to reveal the role of the observation time scale in shaping the leading deterministic dynamics. Upon the basis of the insights thereby generated, for each system, we introduce a suitable measure of control efficiency. Subsequently, we address the influence of the variance of noise, for different observation time windows and different degrees of control response, on the efficiency of both systems under consideration. Finally, in Sec.III we summarize our main results and conclusions.

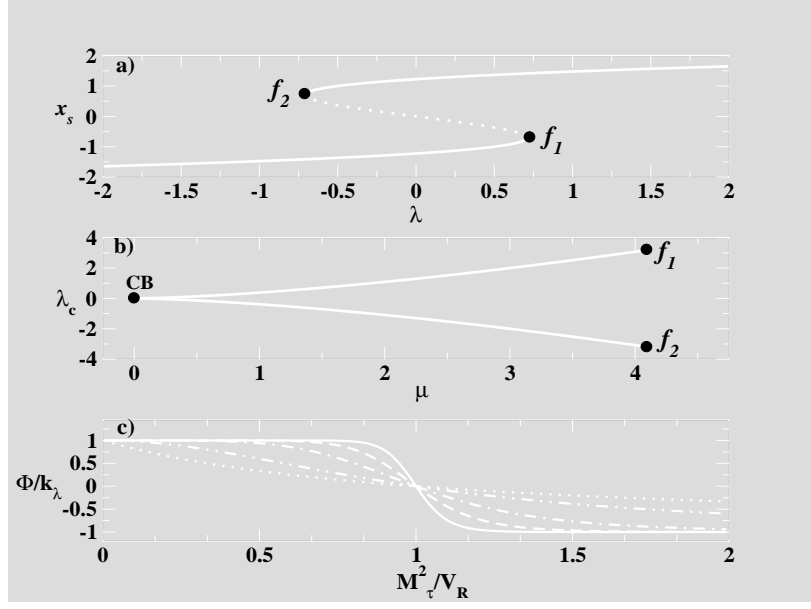


FIG. 1. a) plot of a fold bifurcation diagram, as provided by the stationary solutions of the cusp normal form 5, as a function of parameter λ and for $\mu = 3/2$. The solid lines correspond to the stable branches and the dashed line to the unstable one. Unstable and stable branches meet and annihilate each other at the saddle-node bifurcation points f_1 and f_2 ; b) plot of the branches formed by the critical values of λ corresponding to both f_1 and f_2 bifurcation points, as a function of a second parameter μ . Both branches converge to the cusp bifurcation point CB at $\mu = 0$. Finally, plot Φ/k_λ vs M_τ^2/V_R , for $\alpha = 1$ (dot), $\alpha = 2$ (dot-dot-dash), $\alpha = 5$ (dot-dash), $\alpha = 10$ (dash) and $\alpha = 20$ (solid).

II. NORMAL FORMS AND CONTROL

Regardless of the complexity of its dynamics, as a system evolves further into the vicinity of a bifurcation point, there is a decrease in the number of degrees of freedom necessary to effectively describe the dynamics. This remarkable fact can be understood as a direct consequence of the gradual run out of the fast-relaxation dynamic modes that occurs along with the critical slowing down. A corner stone in local bifurcation theory is the application of the centre manifold method [11] to identifying canonical descriptions of the nonlinear behavior around a bifurcation point, in terms of a reduced number of variables that contain the whole information about the dynamics of the slow leading modes. Such simple representations –so called normal forms– describe the long-term behavior of the normalized amplitude of the stable solutions that emerge at criticality. The importance of normal forms lies on the fact that the dynamics of different types of systems,

around the critical point, can be mapped one-to-one continuously onto the one displayed by a normal form [22]. This property –a property known as topological equivalence– allows to define classes of universality, where near the critical point, the dynamics of systems belonging to the same class can be mapped onto each other by a change of variables preserving the direction of trajectories. Naturally, a simple and most representative element in a class of universality is thus the corresponding normal form.

With the aim of tackling questions *i*)-*iii*) above in a general context, we shall focus our analysis of control dynamics on selected normal forms. As an **EWI** we consider the variance $M_\tau^2[\delta x_j] = \frac{1}{\tau} \int_{t-\tau}^t (\delta x_j(t'))^2 dt'$ of deviations of a system variable around its mean value, within the time window τ , i.e. $\delta x_j = x_j - \frac{1}{\tau} \int_{t-\tau}^t x_j(t') dt'$. Consistently with the minimal character of the normal form description, let us consider a simple expression for the control functional in (2), of the form

$$\Phi(\tau, \alpha) = k_\lambda \left(1 - 2 \frac{M_\tau^2[\delta x_j]^\alpha}{M_\tau^2[\delta x_j]^\alpha + V_R^\alpha} \right), \quad k_\lambda, \alpha > 0 \quad (3)$$

where k_λ corresponds to the maximum exploitation and recovery rates. For low values in α both the exploitation and recovery phases develop weakly as $M_\tau^2[\delta x_j]$ crosses the threshold value V_R . As α is arbitrarily increased, the functional Φ sharply approaches a Heavyside function (see (Fig.(1)c)) –which emulates a full blown response of the control system (2) to a threshold overshooting

$$\Phi(\tau, \alpha \rightarrow \infty) = \begin{cases} k_\lambda & \frac{V_R}{M_\tau^2[\delta x_j]} > 1 \\ 0 & \frac{V_R}{M_\tau^2[\delta x_j]} = 1 \\ -k_\lambda & \text{otherwise} \end{cases} \quad (4)$$

Here, a comment is in order with regard to the choice of the variance of a system variable as an **EWI**. As discussed in [23], in the more general case where the standard deviation of the random perturbations is a time dependent function, the variance of a system variable may not constitute a reliable **EWI**. Consequently, we shall restrain ourselves to the case where both standard deviations ω_x and ω_λ in (1)-(2) are constants.

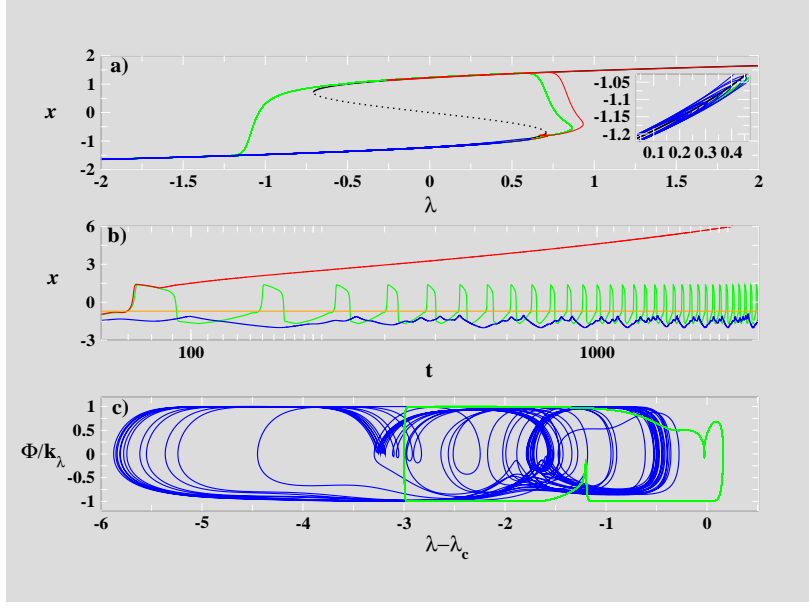


FIG. 2. For $\tau = 10$ (red), $\tau = 11$ (green) and $\tau = 35$ (blue), a) plot of a CCS deterministic trajectories ($\omega_x = 0, \omega_\lambda = 0$) in the (λ, x) -plane (in black lines the bifurcation diagram of the cusp normal form); b) plot x vs time (in orange the critical value x_c) and c) Plot of Φ/k_λ vs $\lambda - \lambda_c$, as related to the system trajectories in panels a) and b), for $\tau = 11$ and $\tau = 35$. Here, values $\Phi > 0$ and $\Phi < 0$ are considered as exploitation and recovery, respectively. Parameters and initial condition: $\mu = 3/2$, $k_\lambda = 1/10$, $\alpha = 2$, $V_R = 1/100$. $x(0) = -3/2$, $\lambda(0) = -5$. Time step $dt = 5 \times 10^{-3}$, numerical integration over 5×10^5 time steps.

A. The cusp-control system (CCS)

As pointed out in [12, 24], the phenomenology observed in so called catastrophic regime shifts in different types of systems can be suitably assimilated with a saddle-node bifurcation. As a parameter evolves, a saddle-node bifurcation occurs when a stable (node) and an unstable (saddle) fixed point solution, to the deterministic evolution law of the system, approach each other and annihilate upon meeting. This entails the destabilization of the system, which is thus led to an abrupt runaway –this constitutes a catastrophic regime shift. However, since in real systems unbounded runaways are ruled out, saddle-node bifurcations are typically found in cases where the branch formed by the system fixed points folds upon itself, within a range of values in the control parameter λ (see Fig.1a). The presence of such a folding entails the existence of a region where two stable solution branches coalesce along with an unstable one. In this picture, the unstable branch marks the boundary line between the two basins of attraction, associated to both stable branches. In general, the

distance between the bifurcation points at the extremes of the s-shape folding (f_1 and f_2 in Fig.1b) is determined by a second parameter μ . Examples of such a bistability are reported in different types of systems, ranging from laser and cell division, to ecosystems and climate [12].

In order to explore the question of bifurcation control in the context of bistability, let us consider the cusp normal form

$$F(x, \lambda) = x(\mu \pm x^2) + \lambda \quad (5)$$

The solutions to the stationary form $F(x_s, \lambda) = 0$ (with the sign minus in (5)), as a function of λ , correspond to those plotted in Fig.1a. The position of the saddle-node bifurcation points $f_{1,2}$ are determined by the relation $f_{1,2} = \{(\lambda, \mu) | \lambda = \pm \frac{2}{3\sqrt{3}} \mu^{3/2}\}$. Accordingly, both branches approach each other and merge as μ tends to zero, at the so-called cusp bifurcation point (Fig.1b). A similar behavior occurs when changing the sign in the second term in (5), which amounts to inverting the s-shaped folding.

We carry out the numerical integration of the variance-based cusp-control system (**CCS**) (1), (2), (3 and 5) via a standard Euler-Maruyama approximation [10].

1. CCS deterministic dynamics

In order to illustrate the dynamics of the **CCS** in absence of noise ($\omega_x = 0, \omega_\lambda = 0$), let us consider exploitation as initially occurring along the negative branch in Fig.1a, a moderate degree of control response ($\alpha = 5$) (see Fig.1c) and a small variance reference threshold $V_R = 1/100$.

According to the numerical results, we distinguish three basic long-term **CCS** behaviors:

- (B1) *Exploitation runaway*: If the time window is small as compared to a critical value τ_c , the system crosses the bifurcation point f_1 and a regime shift occurs. As the system is expelled onto the upper branch, the control mechanism reacts to the overshooting of the reference value V_R . This triggers a recovery stage ($\Phi < 0$) along the upper branch. However, as the system evolves along the upper branch, it eventually loses completely the “memory” on the original low branch regime. As a result, the variance decreases gradually below V_R , recovering is interrupted and the **CCS** switches to runaway exploitation (red lines in Figs.2a,b).
- (B2) *Hysteretic pathways*: When the observation window is now greater than the critical value τ_c , a transition occurs, from runaway exploitation to long-term control. For specific τ -ranges,

the system crosses the bifurcation point f_1 and it starts recovering along the upper branch. However, differently from **B1**, the upper branch recovering continues further and the bifurcation point f_2 is crossed. Thus, the system undergoes again a regime shift towards the original lower branch. As the system stability increases and information about the upper branch is lost, the **CCS** eventually switches to exploitation, beyond f_1 (green lines in Figs.2a,b). This exploitation-recovery process is maintained in the long-term. Processes describing cycles along the zone of bistability are commonly referred as hysteretic. Figure 2c depicts the long-term hysteretic exploitation-recovery process, as a function of the proximity to the critical value λ_c corresponding to the bifurcation point f_1 .

(B3) Branch-confined pathways: For large τ values, escapes to the upper branch are suppressed and the system remains oscillating aperiodically without crossing the bifurcation point f_1 (blue line, Figs.2a,b). This aperiodic dynamics is predictable in the sense that the average growth of an initially small deviation around a reference state [1] is sub-exponential (not shown). The increase in complexity of the dynamics occurring along with τ is illustrated by the inset of Fig.2a and by Fig.2c (blue lines). The overall range of exploitation and recovery becomes larger as τ is increased Fig.2c. In contrast with case **B1**, the evolution towards the bifurcation point is marked by short exploitation-recovery-exploitation cycles Fig.2c, where effective recovery avoids a system regime shift. For certain τ -ranges long-term intermittence between behaviors **B2** and **B3** arises (not shown).

It is worth noticing from Fig.2a that the regime shifts occur past and not at the crossing of the critical point (x_c, λ_c) –following apparent extensions of the lower stable branch, beyond the critical point. Such an “inertia” effect is a result of the critical slowing down, where the dynamics of x becomes ‘slaved’ by the slower dynamics of λ .

2. CCS control efficiency

A meaningful way to characterize further the **CCS** dynamics is to quantify the control capacity to increase and maintain long-term exploitation within a selected stable branch, as a function of the model parameters. Accordingly, we quantify the efficiency as

$$\varepsilon(\tau, \alpha) = \lim_{T \rightarrow \infty} \frac{\omega_R^2}{TM_T^2} \int_0^T \frac{\Phi(\tau, \alpha) H(\Phi(\tau, \alpha)) H(-x(t))}{k_\lambda} dt \quad (6)$$

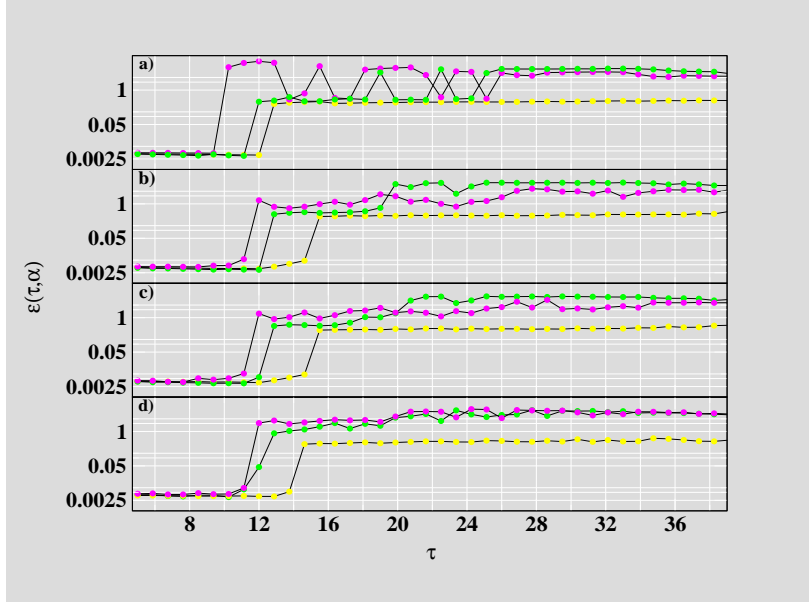


FIG. 3. For $\alpha = 1$ (yellow), $\alpha = 5$ (green) and $\alpha = 20$ (magenta) and $\omega_\lambda = 0$, a) plot of ε vs τ for deterministic **CCS** trajectories ($\omega_x = 0$). Similar plots for stochastic trajectories with b) $\omega_x = 5$, c) $\omega_x = 10$ and d) $\omega_x = 25$. Parameter values, initial condition and numerical integration as in Fig.2. Here, each ε value has been computed for 1×10^6 time steps.

H denotes a Heavyside function, whose value is zero for negative argument or one otherwise. The product of the Heavyside functions in the integral plays the role of an AND boolean operator, entailing that the contributions to the time average of Φ are considered only if Φ is positive and x negative. The variance of deviations from the initial to the final time T , in the denominator of (6), aims at amplifying differences between hysteretic (**B2**) and branch-confined (**B3**) control regimes. Without loss of generality, hereafter we set the reference value $\omega_R^2 = 1$ in (6).

In the next subsections we focus on the behavior of the efficiency as a function of the size of the observation window τ , of the degree of control response α and of the variance of random fluctuations ω_x and ω_λ .

3. Deterministic CCS efficiency

Figure 3a summarizes the behavior of the efficiency (6) over the deterministic **CCS** trajectories corresponding to different values in τ and α . Efficient control rises sharply as the observation window is increased above an α -specific threshold τ_c —since below this value the **CCS** undergoes

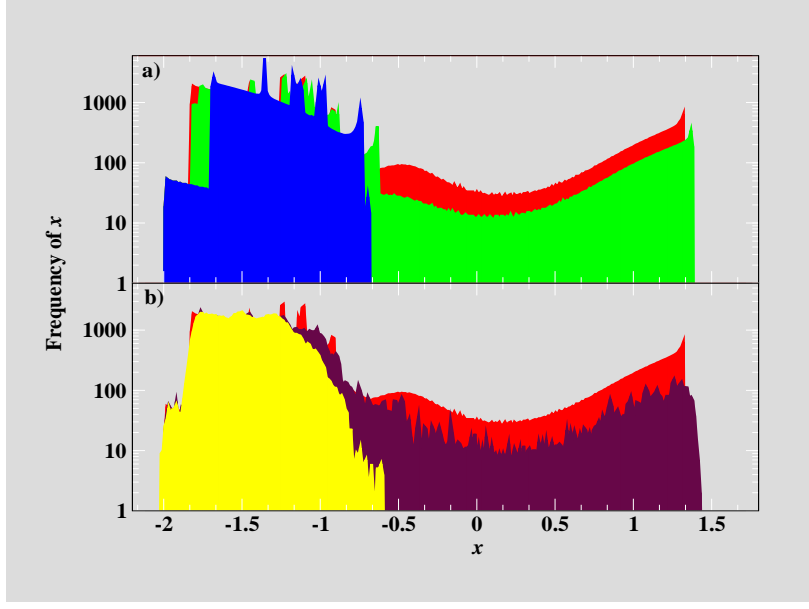


FIG. 4. For $\alpha = 20$, a) frequency histograms of **CCS** deterministic trajectories ($\omega_x = 0$, $\omega_\lambda = 0$) for $\tau = 22.5$ (red), $\tau = 22$ (green) and $\tau = 19.5$ (blue). For $\tau = 22.5$ and $\omega_\lambda = 0$, b) histograms of stochastic trajectories with $\omega_x = 10$ (brown) and $\omega_x = 25$ (yellow), as well as histogram of the deterministic case (red), as in panel a). Parameter values, initial condition and numerical integration as in Fig.(2).

exploitation runaway (**B1**). Above τ_c , the behavior of the efficiency becomes highly dependent on the degree of control response. For a low control response ($\alpha = 1$), the efficiency exhibits a quasi-constant plateau for $\tau > \tau_c$. This plateau consists of hysteretic cases (**B2**). In cases of moderate ($\alpha = 5$) and high ($\alpha = 20$) degree of control response, the behavior of the **CCS** efficiency becomes non-trivial. For values $\tau \gg \tau_c$ the efficiency is remarkably enhanced (see the appearance of a second plateau occurring for $\tau \geq 25$ in Fig3a), as a result of the suppression of the **CCS** hysteretic control pathways, in favour of branch-confined recovery-exploitation cycles (**B3**). For intermediate values $\tau > \tau_c$, a combination of pathways (**B2**) and (**B3**) is observed –where low efficiency values correspond to hysteretic control pathways (**B2**). Such a τ -dependent selection of control pathways is further illustrated by the histogram of the x variable in Fig.4a, for the high control response case. It shows that the dynamics of the **CCS** becomes confined to $x < 0$ values as τ is slightly shifted, which explains the alternance of low and high efficiency ranges appearing at intermediate τ values. Comparing the different control response cases in Fig.3a, it can be observed that the moderate one is the most efficient for $\tau \gg \tau_c$.

4. Stochastic CCS efficiency

Let us address now the case where the evolution of x is marked by continuous random perturbations (i.e., $\omega_x \neq 0$, $\omega_\lambda = 0$ in Eqs. (1) and (2)). Panels b) to d) in Fig.(3) illustrate the influence of noise on the **CCS** efficiency. In the case of low control response, the behavior of the efficiency with τ appears to be independent of the standard deviation of noise ω_x . In contrast, for moderate and high control responses, the overall efficiency becomes enhanced by an increase in the intensity of noise, in the sense that low efficiency values tend to disappear while high efficiency occurs all the way above the τ_c -value (compare Fig.3a with Figs. 3b–d). However, if the variance of noise is marginally large (Fig.3a-c), the high control response case is less efficient as compared with the moderate one. For strong noise (Fig.3d), both moderate and high control response cases converge to a similar efficiency value for large τ .

Since, for high and moderate response cases, low efficiency values tend to disappear when increasing intensity noise, it is natural to inquire about the influence of strong noise on hystheretic pathways. This effect is shown by the histogram of x Fig.4b for a single realization with $\alpha = 20$. Notice the similarity between panels (a) and (b), which illustrates the equivalence in the control pathway selection that occurs either by changes in τ (Fig.4a) or by an increase in the standard deviation of the random fluctuations ω_x (Fig.4b).

Interestingly enough, we observe a similar effect of efficiency enhancement and uniformization in the case of noise at the level of the control system (i.e., $\omega_x = 0$, $\omega_\lambda \neq 0$) Fig.5a-c, as compared with the deterministic **CCS** (Fig.3a). However, in this case the emergence of efficient control occurs for larger values in τ , as the intensity of noise increases. Finally, in the case of strong noise at the level of both the state variable x and control (i.e., $\omega_x \gg 0$, $\omega_\lambda \gg 0$) Fig.5d, the case of high control response becomes the most efficient one, in comparison with the ($\omega_x \neq 0$, $\omega_\lambda = 0$) cases (Fig.3d).

B. The Hopf-control system (HCS)

A relevant and also frequently observed type of regime shift consists in the onset of time periodic behavior, as a stationary system regime becomes unstable at a threshold value in a system parameter. Examples of processes and systems where such phenomenon is present are for instance, chemical clocks at the mesoscopic [25] and macroscopic [26] scales, cellular biochemical cycles [27], prey-predator population dynamics [28], shallow lake ecosystems [7], the ocean-atmosphere

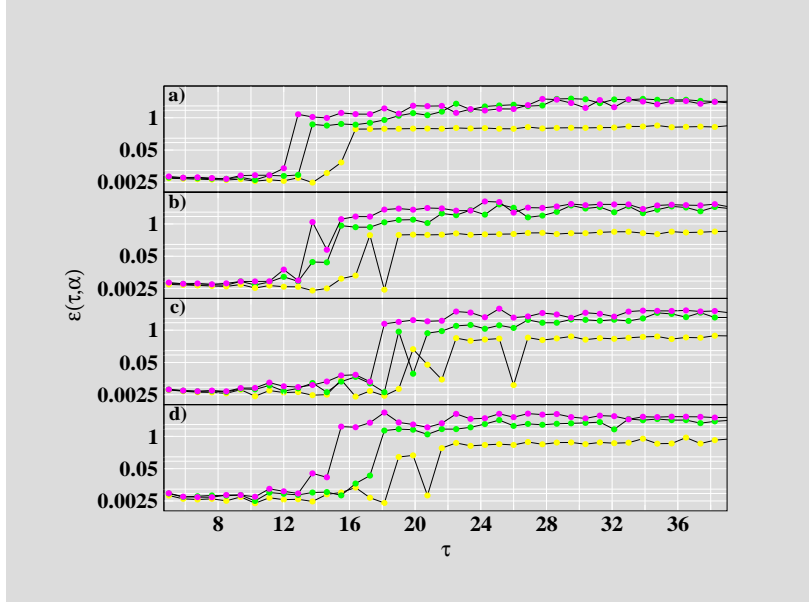


FIG. 5. For $\alpha = 1$ (yellow), $\alpha = 5$ (green) $\alpha = 20$ (magenta) and $\omega_x = 0$, plot of ε vs τ for stochastic CCS trajectories with a) $\omega_\lambda = 5$, b) $\omega_\lambda = 10$ and c) $\omega_\lambda = 20$. In d), similar plots for $\omega_\lambda = 25$ and $\omega_x = 20$. Parameter values, initial condition and numerical integration as in Fig.2. Here, each ε value has been computed for 1×10^6 time steps.

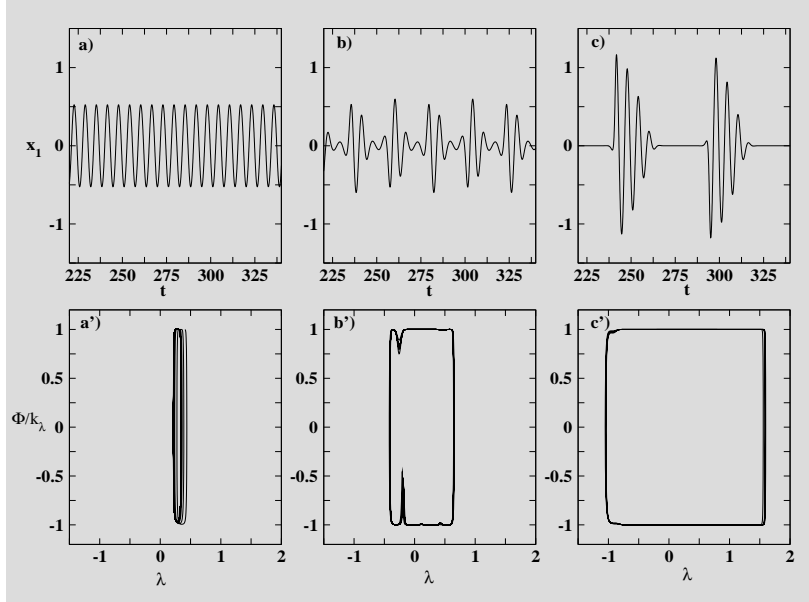


FIG. 6. Behavior of the deterministic HCS ($\omega_x = 0, \omega_\lambda = 0$). Plot of x_1 vs time for a) $\tau = 0.5$, b) $\tau = 3$, c) $\tau = 10$. Corresponding to these τ values, panels a')-c') show the plot Φ/k_λ vs λ .

system [29], or non-equilibrium economics models [30]. All models accounting for the emergence of oscillatory behavior involve at least two variables and in all cases it has been possible to assimilate the mechanisms underlying the critical onset of oscillations to a Hopf bifurcation [11]. In what follows we focus on the dynamics of the coupled system of control (2)-(3) and the deterministic evolution law in (1), given by the Hopf normal form –which we refer hereafter as the Hopf-control system (**HCS**). For illustrative purposes and without loss of generality, we consider the supercritical version of the Hopf normal form

$$F_1(x_1, x_2, \lambda) = \lambda x_1 - x_2 - x_1(x_1^2 + x_2^2), \quad F_2(x_1, x_2, \lambda) = \lambda x_2 + x_1 - x_2(x_1^2 + x_2^2) \quad (7)$$

The stationary solution $x_1 = 0, x_2 = 0$ to the Hopf normal form (7) loses stability as λ approaches zero along the negative axis. Above the critical value $\lambda_c = 0$, the system variables (x_1, x_2) describe stable oscillations whose amplitudes increase as $\sqrt{\lambda}$.

Here, the control functional (3) is fed by either of both system variables (x_1, x_2) , whose dynamics is subjected to stochastic perturbations with same standard deviation ω_x .

1. *Deterministic HCS dynamics*

For an initial condition around the stationary solution $x_1 = 0, x_2 = 0$ and for an initial value $\lambda < 0$, Figs.6a-c show the deterministic dynamics of the x_1 component under the influence of control for different values of the time window τ . In complete absence of control, the linear growth in the value of λ , beyond the critical point, leads to oscillations of increasing amplitude $\sqrt{\lambda}$. In contrast, for small values $\tau \gtrsim 0$ the amplitude of oscillations becomes constant (Fig.6a). In this situation, both recovery ($\Phi < 0$) and exploitation ($\Phi > 0$) phases succeed each other rapidly, while the control variable always remains above the critical value $\lambda_c = 0$ (Fig. 6a'). A further increase in τ induces a temporal alternance of small and large oscillation amplitudes (Fig.6b), as the recovery and exploitation phases lead the system back and forth across λ_c (Fig.6b'). If τ is large enough, oscillations are temporally suppressed (Fig.6c), during a time interval that increases with τ . This situation results from the fact that the λ -interval covered by the recovery and exploitation phases becomes enlarged (Fig.6c'). In contrast with the **CCS** Fig.5c, no substantial increase in the complexity of the dynamics is observed as the time window τ is arbitrarily increased.

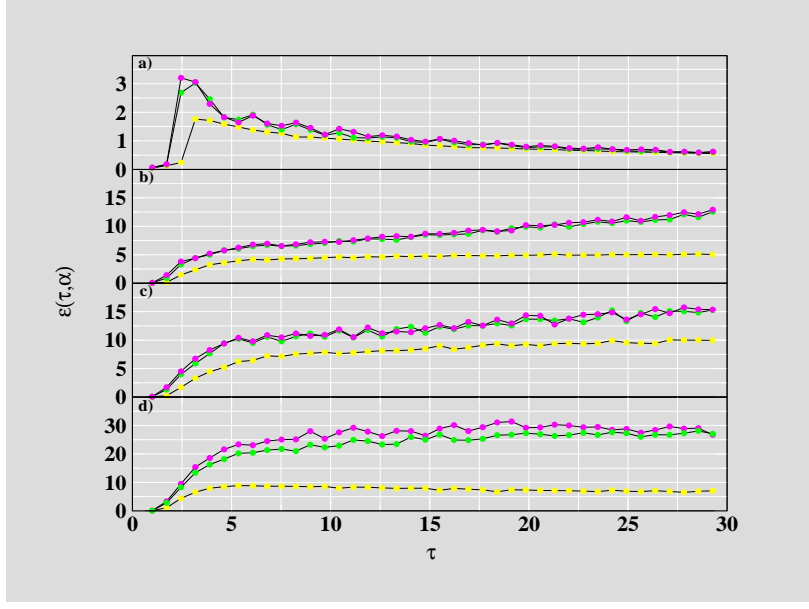


FIG. 7. For $\alpha = 1$ (yellow), $\alpha = 5$ (green), $\alpha = 20$ (magenta) and $\omega_\lambda = 0$, a) plot of ε vs τ for deterministic **HCS** trajectories ($\omega_x = 0$). Similar plots for stochastic trajectories with b) $\omega_x = 5$, c) $\omega_x = 10$ and d) $\omega_x = 25$. Parameter values, initial condition and numerical integration as in Fig.2. Here, each ε value has been computed for 1×10^6 time steps.

2. Deterministic and stochastic HCS efficiency

In a similar vein as for the **CCS**, we define the efficiency of the **HCS** as the capacity to increase and maintain long-term subcritical exploitation, while keeping the oscillation amplitudes to a minimum. Thus we write the efficiency as

$$\varepsilon(\tau, \alpha) = \lim_{T \rightarrow \infty} \frac{\omega_R^2}{TM_T^2} \int_0^T \frac{\Phi(\tau, \alpha) H(\Phi(\tau, \alpha)) H(-\lambda(t))}{k_\lambda} dt \quad (8)$$

where H denotes a Heavyside function, such as defined for (6).

As shown by Figs.6a'–c', subcritical exploitation increases with the size of the observation window τ . However, it is also observed that the amplitude of oscillations along the recovery phase also increases with τ (Figs.6a–c). Therefore, an optimum is expected to occur in the efficiency (8) as a function of τ . This situation is actually illustrated in Fig.7a for different degrees of control response. Notice that compared to the efficiency of the deterministic **CCS** Fig.6a, the sharp rise towards efficient control occurs at a small value τ_c , above which efficiency gradually decays for larger τ values.

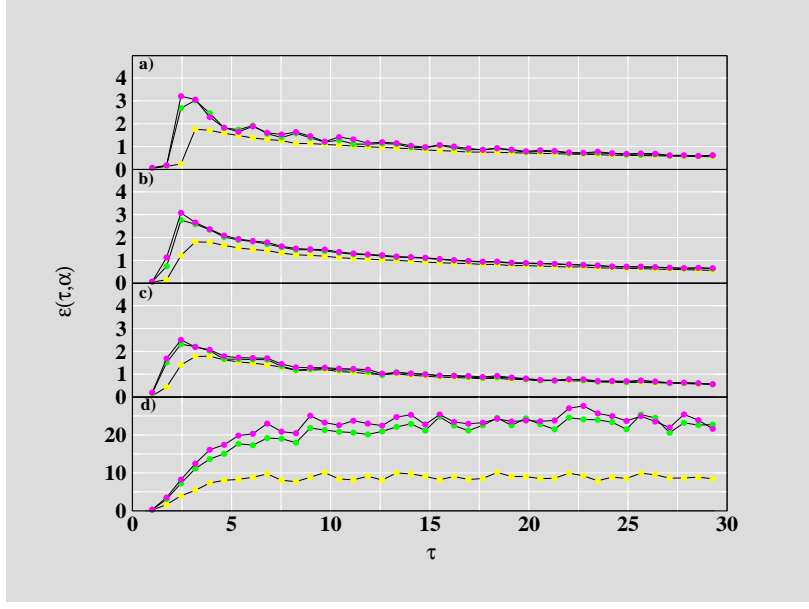


FIG. 8. For $\alpha = 1$ (yellow), $\alpha = 5$ (green) $\alpha = 20$ (magenta) and ($\omega_x = 0$), plot of ε vs τ for stochastic HCS trajectories with a) $\omega_\lambda = 5$, b) $\omega_\lambda = 10$ and c) $\omega_\lambda = 20$. In d), similar plots for $\omega_\lambda = 25$ and $\omega_x = 20$. Parameter values, initial condition and numerical integration as in Fig.2.

In presence of noise at the level of the system variables (x_1, x_2) (i.e. $\omega_x \neq 0, \omega_\lambda = 0$), it is observed that the efficiency increases along with τ , for the moderate and high control response cases, while for low control response it remains at low practically constant value (Fig.7b). Similarly as for the CCS (Fig.3), the efficiency is in overall enhanced by the intensity of noise Fig.7b–d. Moreover, the behavior of the efficiency is practically the same for moderate and high control response cases, with the exception of the strong noise case Fig.7d, where high control response is highest for arbitrary large τ values.

When considering the effect of stochastic perturbations on the control dynamics (i.e. $\omega_x = 0, \omega_\lambda \neq 0$) Figs.8a–c, the behavior of the efficiency remains qualitatively the same as in the purely deterministic case Fig.7a. Regardless the intensity of control fluctuations, a maximum in efficiency is observed at an approximately same τ value. The efficiency associated with high and moderate control responses exhibit a similar dependency on τ . In both cases, the maximum in efficiency is higher as compared to the low response one. However, the highest efficiency peaks are slightly lowered when augmenting the intensity of noise at the level of control. For all the control response cases, when increasing τ the corresponding efficiencies tend to converge to a same low value,

independently of the intensity of noise in control.

Finally, in presence of strong fluctuations at the level of the dynamics of both the system and the control (i.e. $\omega_x \gg 0, \omega_\lambda \gg 0$), the efficiency rises towards a plateau for larger values of τ . In this case it is observed that the presence of strong fluctuations enhances the control efficiency and to a greater extent in the cases of high and moderate control response.

III. SUMMARY AND CONCLUSIONS

In this work we addressed the variance-based control of systems evolving towards criticality. As testbed models we considered the cusp and Hopf normal forms, displaying the prototypical phenomena of bistability and oscillations, respectively.

In relation to the core questions *i)–iii)* exposed in Sec. I, our main results can be summarized as follows:

i) In absence of noise, effective control in both the cusp and the Hopf systems rises above a threshold value in the size of the observation time window.

ii) In the cusp-control system, changes in the observation time window provide a means to selecting among different control pathways, namely, hysteretic, branch-confined control, as well as an intermittent combination of both. The branch-confined regime becomes dominant when considering a large observation window. This entails an increase in the complexity of the dynamics, in the form of aperiodic control-system trajectories. In the case of the Hopf-control system, the crossing of the bifurcation point could not be avoided, for the range of parameters explored. Instead, as the observation time window is enlarged, exploitation and recovery phases occur for longer periods. This gives rise to the alternance of time intervals where oscillations are suppressed, followed by amplified oscillations.

iii) In both systems, random fluctuations, at the level of the system variables, enhance the efficiency of control, and especially for large observation time windows. In the case of the cusp-control system, the intensity of noise induces a control pathway selection, similarly as in the deterministic case when varying the observation time window. From the deterministic to the strong fluctuations cases, the efficiency of the cusp-control system tends to increase for large values in the size of the observation window. In the Hopf-control system, an increase in the intensity of noise induces a reduction in the amplitude of oscillations, which entails an increase in the control efficiency. Random

fluctuations at the level of the control dynamics exert opposite influences on the control efficiency of the cusp and Hopf control systems. In the former it increases, while for the latter it tends to diminish the peak of efficiency. For both cusp and Hopf control systems, the combination of fluctuations in the control and in the system dynamics amplifies the efficiency in particular for large observation time windows. Regarding the interplay between the efficiency, the degree of control response, the intensity of noise, and the observation time window, we observe: in both systems, the low control response case exhibits a low efficiency, independently of the intensity of noise and observation window. When increasing the variance of fluctuations in the dynamics of the cusp system, a moderate control response leads to a higher efficiency; the opposite situation occurs when considering fluctuations on the control dynamics. In the Hopf system, the moderate and high control response perform very similar, regardless the intensity of noise and size of the time window. In this situation, for both moderate and high response cases, the efficiency converges to the lower value associated with a low control response, as the observation time window is increased.

For the systems here addressed, it remains to be studied the influence of other parameters, such as the rate of change in the control system (k_λ), the control reference parameter (V_R), as well as alternative approaches towards bifurcation, such as variation of the parameter μ in the cusp map—which would induce flickering patterns at the vicinity of the critical point.

Potential applications of this approach to real world systems include ecosystem models, non-equilibrium mesoscopic physico-chemical systems or non-equilibrium models in economics, among others. Similarly, it is of interest to extend this work to address the control properties associated to different alternative early warning indicators. Moreover, our analysis could be extended to address the control of coupled systems, forming networks whose nodes consist in dynamical systems subject to control.

Certainly, the structure of the evolution equations of real world models, involving a large number of parameters, lacks the degree of symmetry that confers to normal forms their characteristic simplicity. Moreover, not every dynamical system exhibiting a bifurcation belongs to a class of universality. It is thus important to address further the relation between the observation time scales leading to effective control and the geometrical properties, such as the curvature, of the system solution branches, while considering different early warning indicators.

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